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1978 J. Phys. A: Math. Gen. 11 687

(http://iopscience.iop.org/0305-4470/11/4/009)

J. Phys. A: Math. Gen., Vol. 11, No. 4, 1978. Printed in Great Britain. © 1978

Electrons in logarithmic potentials II. Solution of the Dirac equation

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Received 12 August 1977, in final form 3 November 1977

Abstract. We derive the asymptotic behaviour of the general solution of Dirac's equation for an electron in a logarithmic potential from stability theorems and investigate the energy spectrum, thereby recognising significant differences in comparison with corresponding properties of the Schrödinger equation. An uniformly convergent perturbation expansion of the general solution is presented, in analogy to our treatment of the Schrödinger equation, which may be used to compute biprisma interferences of electron waves.

1. Introduction

In our first paper (Gesztesy and Pittner 1978, to be referred to as I) on logarithmic potentials we have solved each radial Schrödinger equation by an uniformly convergent perturbation expansion. Since the electrons are accelerated to relativistic energies in the interference experiments (Möllenstedt and Düker 1956, Donati *et al* 1973, Merli *et al* 1976) quoted in § 1 of I, we shall now investigate the Dirac equation. This attempt leads to the recognition of significant differences between corresponding properties of the Schrödinger and Dirac equations which we shall at first describe qualitatively.

Due to the electron spin each of the coupled radial equations resulting from the separation of variables in the Dirac equation is in the limit point case near zero, as can be seen from the asymptotic behaviour near zero of its general solution; the limit circle case does not occur here, in contrast to the Schrödinger equation. The oscillating nature of the general asymptotic solution near infinity forbids discrete eigenvalues of the Dirac operator. The essential spectrum of each radial operator is continuous and covers the whole real line, both in the attractive and repulsive cases, due to the possible appearance of positrons.

At first the general solution of each radial Dirac equation is approximated asymptotically near zero and infinity, and then it is represented in terms of the uniformly convergent perturbation expansion which corresponds to our rigorous solution of the Schrödinger equation.

2. Self-adjointness and angular momentum expansion

The formal differential operator

$$T = \alpha \cdot \nabla/\mathrm{i} + \beta m + V \tag{2.1}$$

† Supported by Fonds zur Förderung der wissenschaftlichen Forschung in Österreich, Projekt Nr. 3225.

with the usual Dirac matrices α, β (for our notation see Bjorken and Drell 1964) and the logarithmic potential

$$V(r)=\epsilon\ln(r/b), \qquad r=[(x^1)^2+(x^2)^2]^{1/2}>0, \tag{2.2}$$

may be transformed to cylindrical coordinates r, ϕ, x^3 via the notations

$$\sqrt{2}\alpha^{\pm}=\alpha^1\pm\mathrm{i}\alpha^2, \qquad \sqrt{2}D_{\pm}=\frac{\partial}{\mathrm{i}\,\partial x^1}\pm\frac{\partial}{\partial x^2}=\mathrm{e}^{\pm\mathrm{i}\phi}\left(\frac{\partial}{\mathrm{i}\,\partial r}\pm\frac{\partial}{r\,\partial\phi}\right), \tag{2.3}$$

the insertion of which yields

$$T=\alpha^+D_-+\alpha^-D_++\alpha^3\frac{\partial}{\mathrm{i}\,\partial x^3}+\beta m+V. \tag{2.4}$$

The logarithmic potential is relatively bounded with respect to the free Dirac operator in the Hilbert space $\{L_2(\mathbb{R}^3)\}^4$ with relative bound smaller than one, because of $V\in L_3^{\mathrm{loc}}(\mathbb{R}^3)$, and therefore the following statement holds (Jörgens 1973).

Theorem 2.1. The restriction $T|\{C_0^\infty(\mathbb{R}^3)\}^4$ is essentially self-adjoint.

On Dirac spinors of the product form

$$\begin{aligned}\Psi_l^{(+)}(r,\phi)&=r^{-1/2}\begin{bmatrix}\mathrm{e}^{\mathrm{i}l\phi}g_{0,l}^{(+)}(r)\\ 0\\ 0\\ \mathrm{i}\,\mathrm{e}^{\mathrm{i}(l+1)\phi}g_{1,l}^{(+)}(r)\end{bmatrix},\\ \Psi_l^{(-)}(r,\phi)&=r^{-1/2}\begin{bmatrix}0\\ \mathrm{i}\,\mathrm{e}^{\mathrm{i}(l+1)\phi}g_{1,l}^{(-)}(r)\\ \mathrm{e}^{\mathrm{i}l\phi}g_{0,l}^{(-)}(r)\\ 0\end{bmatrix},\end{aligned} \tag{2.5}$$

the formal differential operators describing the projections of total angular momentum and spin perpendicular to the plane of motion ($x^3=0$) act in the following way:

$$\begin{aligned}\left(\frac{\partial}{\mathrm{i}\,\partial\phi}+\tfrac{1}{2}\sigma^3\right)\Psi_l^{(\pm)}(r,\phi)&=(l+\tfrac{1}{2})\Psi_l^{(\pm)}(r,\phi),\\ \beta\sigma^3\left(\frac{\partial}{\mathrm{i}\,\partial\phi}+\tfrac{1}{2}\sigma^3\right)\Psi_l^{(\pm)}(r,\phi)&=\pm(l+\tfrac{1}{2})\Psi_l^{(\pm)}(r,\phi),\end{aligned} \tag{2.6}$$

with any integer number l. The formal energy eigenvalue equation

$$T\Psi_l^{(\pm)}=E\Psi_l^{(\pm)} \tag{2.7}$$

is decomposed into the two ordinary matrix differential equations

$$\begin{aligned}&\begin{bmatrix}\dfrac{\mathrm{d}}{\mathrm{d}r}-(l+\tfrac{1}{2})(r^{-1}) & m-V(r)+E\\ m+V(r)-E & \dfrac{\mathrm{d}}{\mathrm{d}r}+(l+\tfrac{1}{2})(r^{-1})\end{bmatrix}\begin{bmatrix}g_{0,l}^{(+)}(r)\\ g_{1,l}^{(+)}(r)\end{bmatrix}=0,\\ &\begin{bmatrix}\dfrac{\mathrm{d}}{\mathrm{d}r}+(l+\tfrac{1}{2})(r^{-1}) & -m+V(r)-E\\ -m-V(r)+E & \dfrac{\mathrm{d}}{\mathrm{d}r}-(l+\tfrac{1}{2})(r^{-1})\end{bmatrix}\begin{bmatrix}g_{1,l}^{(-)}(r)\\ g_{0,l}^{(-)}(r)\end{bmatrix}=0,\end{aligned} \qquad l=0,\pm1,\pm2,\ldots. \tag{2.8}$$

Each of these radial equations is in the limit point case near zero and infinity, as can be learned from the asymptotic behaviour of their solutions to be derived in the next section. For any integer l there are solutions $g_{i,l}^{(\pm)}(r)$, $i=0, 1$, which are square-integrable near zero, and others which are not of this type. Due to the electron spin the limit circle case does not show up here, because the total angular momentum never vanishes.

3. Asymptotic solutions and spectral properties

With the notations

$$G_l^{(+)}(r)=\begin{bmatrix} g_{0,l}^{(+)}(r) \\ g_{1,l}^{(+)}(r) \end{bmatrix}, \qquad G_l^{(-)}(r)=\begin{bmatrix} g_{1,l}^{(-)}(r) \\ g_{0,l}^{(-)}(r) \end{bmatrix},$$
$$W_l^{(\pm)}(r)=\pm\begin{bmatrix} j^3/r & V(r)-E-m \\ E-m-V(r) & -j^3/r \end{bmatrix}, \qquad j^3=l+\tfrac{1}{2}, \tag{3.1}$$

our differential equations (2.8) take the matrix form

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)G_l^{(\pm)}(r)=W_l^{(\pm)}(r)G_l^{(\pm)}(r), \qquad l=0, \pm 1, \pm 2, \ldots. \tag{3.2}$$

Aiming firstly at the asymptotic behaviour near zero, we transform

$$G_l^{(\pm)}(r)=Y_l^{(\pm)}(x)=\begin{bmatrix} y_l^{(\pm)}(x) \\ z_l^{(\pm)}(x) \end{bmatrix}, \qquad x=\ln(r/b), \tag{3.3}$$

and obtain the matrix differential equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)Y_l^{(\pm)}(x)=(C_l^{(\pm)}+B^{(\pm)}(x))Y_l^{(\pm)}(x),$$
$$C_l^{(\pm)}=\pm j^3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B^{(\pm)}(x)=\pm\begin{bmatrix} 0 & (\beta x-\lambda)\,\mathrm{e}^x \\ (\mu-\beta x)\,\mathrm{e}^x & 0 \end{bmatrix}, \qquad x\in\mathbb{R}, \tag{3.4}$$

where $\beta=b\epsilon$, $\lambda=b(E+m)$, $\mu=b(E-m)$, the general solution of which is an entire function. Stability theorems about systems of linear differential equations (Coppel 1965) teach us that the general solution behaves asymptotically as

$$Y_l^{(\pm)}(x)\sim c_1\,\mathrm{e}^{\pm j^3x}\begin{bmatrix} 1 \\ 0 \end{bmatrix}+c_2\,\mathrm{e}^{\mp j^3x}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad x\to-\infty, \tag{3.5}$$

or equivalently

$$G_l^{(\pm)}(r)\sim c_1 r^{\pm j^3}\begin{bmatrix} 1 \\ 0 \end{bmatrix}+c_2 r^{\mp j^3}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad r\to 0, \quad j^3=\pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \ldots. \tag{3.6}$$

These two general solutions show the features indicated at the end of the preceding section.

The general solution of equation (3.2) can be approximated asymptotically near infinity via the transformation

$$G_l^{(\pm)}(r)=H_l^{(\pm)}(t), \qquad t=\rho(\ln\rho-1), \qquad \rho=r/b\geqslant 1, \tag{3.7}$$

which leads to the matrix differential equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)H_l^{(\pm)}(t)=A_l^{(\pm)}(t)H_l^{(\pm)}(t), \qquad t\geqslant -1,$$

$$A_l^{(\pm)}(t)=\pm\begin{bmatrix} j^3/(\rho\ \ln\rho) & \beta-\lambda/\ln\rho \\ \mu/\ln\rho-\beta & -j^3/(\rho\ \ln\rho)\end{bmatrix}. \tag{3.8}$$

The limit matrix

$$A^{(\pm)}=\lim_{t\to\infty}A_l^{(\pm)}(t)=\pm\beta\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix} \tag{3.9}$$

possesses the eigenvalues and eigenvectors

$$A^{(\pm)}a_k=\alpha_k^{(\pm)}a_k, \qquad k=1,2,$$
$$\alpha_1^{(\pm)}=-\alpha_2^{(\pm)}=\pm\mathrm{i}|\beta|, \qquad a_1=\begin{bmatrix}1\\ \mathrm{i}\end{bmatrix}, \qquad a_2=\begin{bmatrix}\mathrm{i}\\ 1\end{bmatrix}. \tag{3.10}$$

The characteristic equation

$$\det(A_l^{(\pm)}(t)-\alpha_l^{(\pm)}(t))=0 \tag{3.11}$$

determines the eigenvalues

$$\alpha_{1,l}^{(\pm)}(t)=-\alpha_{2,l}^{(\pm)}(t)=\pm\mathrm{i}w_l(t), \tag{3.12}$$
$$w_l(t)=\{(\beta-\lambda/\ln\rho)(\beta-\mu/\ln\rho)-[j^3/(\rho\ \ln\rho)]^2\}^{1/2},$$

such that

$$\lim_{t\to\infty}w_l(t)=|\beta|, \qquad \lim_{t\to\infty}\alpha_{k,l}^{(\pm)}(t)=\alpha_k^{(\pm)}, \qquad k=1,2. \tag{3.13}$$

Again stability theorems (Coppel 1965) may be applied to derive the general asymptotic solution

$$H_l^{(\pm)}(t)=c_1\exp\left(\pm\mathrm{i}\int_0^t \mathrm{d}\tau w_l(\tau)\right)\left(\begin{bmatrix}1\\ \mathrm{i}\end{bmatrix}+\mathrm{o}(1)\right)$$
$$+c_2\exp\left(\mp\mathrm{i}\int_0^t \mathrm{d}\tau w_l(\tau)\right)\left(\begin{bmatrix}\mathrm{i}\\ 1\end{bmatrix}+\mathrm{o}(1)\right), \qquad t\to\infty, \tag{3.14}$$

or equivalently

$$G_l^{(\pm)}(r)=c_1\exp(\pm\mathrm{i}|\beta|h(r))\left(\begin{bmatrix}1\\ \mathrm{i}\end{bmatrix}+\mathrm{o}(1)\right)+c_2\exp(\mp\mathrm{i}|\beta|h(r))\left(\begin{bmatrix}\mathrm{i}\\ 1\end{bmatrix}+\mathrm{o}(1)\right),$$
$$h(r)=\frac{r}{b}\left[\ln\left(\frac{r}{b}\right)-1-\frac{E}{\epsilon}+\mathrm{o}(1)\right], \qquad r\to\infty. \tag{3.15}$$

Since this general solution is oscillating as $r\to\infty$, no solution at all is square-integrable near infinity.

Since each radial Dirac equation (3.2) is in the limit point case near zero and infinity, the radial operators are self-adjoint in the Hilbert space $\{L_2(\mathbb{R}^+, \mathrm{d}r)\}^2$. The spectra of these operators contain no discrete eigenvalues, according to the asymptotic behaviour near infinity stated above. More precisely, we quote (Weidmann 1971) the following theorem.

Theorem 3.1. The essential spectrum of each radial Dirac operator is continuous and covers the whole real line.

4. Uniformly convergent perturbation expansion

Similarly to our rigorous solution of the Schrödinger equation, we try to expand the general solution of each radial Dirac equation (3.4) in terms of appropriate exponential polynomials, keeping in mind the asymptotic behaviour (3.5) of solutions, and prove the uniform convergence of this perturbation series by an estimate of its coefficients.

Theorem 4.1. The system of regular differential equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\begin{bmatrix} y_l^{(\pm)}(x) \\ z_l^{(\pm)}(x)\end{bmatrix} = \pm\begin{bmatrix} +j^3 & (\beta x-\lambda)\,\mathrm{e}^x \\ (\mu-\beta x)\,\mathrm{e}^x & -j^3\end{bmatrix}\begin{bmatrix} y_l^{(\pm)}(x) \\ z_l^{(\pm)}(x)\end{bmatrix},$$
$$x\in\mathbb{C},\quad j^3=l+\tfrac{1}{2}=\pm\tfrac{1}{2},\ \pm\tfrac{3}{2},\ldots, \tag{4.1}$$

is solved by the convergent series

$$\begin{aligned} y_l^{(\pm)}(x) &= \mathrm{e}^{\pm j^3 x}\sum_{n=0}^{\infty}\mathrm{e}^{2nx}p_{n,l}^{(\pm)}(x),\\ z_l^{(\pm)}(x) &= \mathrm{e}^{\mp j^3 x}\sum_{n=0}^{\infty}\mathrm{e}^{2nx}q_{n,l}^{(\pm)}(x), \qquad x\in\mathbb{C},\end{aligned} \tag{4.2}$$

with polynomials $p_{n,l}^{(\pm)}$ and $q_{n,l}^{(\pm)}$ determined by the recursion scheme which follows from insertion of the series (4.2) into equation (4.1) and comparison with respect to powers of e^{2x}. For $l\geqslant 0$:

$$\begin{aligned}
&p_{n-l,l}^{(+)\prime}(x)+2(n-l)p_{n-l,l}^{(+)}(x)=(\beta x-\lambda)q_{n,l}^{(+)}(x), && n=l,\ l+1,\ldots;\\
&q_{n,l}^{(+)\prime}(x)+2nq_{n,l}^{(+)}(x)=(\mu-\beta x)p_{n-l-1,l}^{(+)}(x), && n=l+1,\ l+2,\ldots;\\
&q_{n,l}^{(+)}(x)=0, \qquad n=0,1,\ldots,l-1; && q_{l,l}^{(+)\prime}(x)+2lq_{l,l}^{(+)}(x)=0.
\end{aligned}$$

For $l<0$: (4.3)

$$\begin{aligned}
&p_{n-l,l}^{(+)\prime}(x)+2(n-l)p_{n-l,l}^{(+)}(x)=(\beta x-\lambda)q_{n,l}^{(+)}(x), && n=0,1,2,\ldots;\\
&q_{n,l}^{(+)\prime}(x)+2nq_{n,l}^{(+)}(x)=(\mu-\beta x)p_{n-l-1,l}^{(+)}(x), && n=0,1,2,\ldots;\\
&p_{n-l,l}^{(+)\prime}(x)+2(n-l)p_{n-l,l}^{(+)}(x)=0, && n=l,\ l+1,\ldots,-1;\\
&p_{n-l-1,l}^{(+)}(x)=0, \qquad n=l+1,\ l+2,\ldots,-1, && \text{if } l\leqslant -2.
\end{aligned}$$

The corresponding recursion scheme for $p_{n,l}^{(-)}$ and $q_{n,l}^{(-)}$ can be obtained from the above one by interchanging $p_{n,l}^{(\pm)}\leftrightarrow q_{n,l}^{(\mp)}$ and $\lambda\leftrightarrow\mu$.

The series (4.2) converges uniformly on each compact subset of $\mathbb{C}$, and it converges uniformly on the negative real line. Therefore, starting with the polynomials

$$\begin{aligned} &p_{0,l}^{(+)}(x)=1 \quad\text{and}\quad q_{l,l}^{(+)}(x)=0 \quad\text{for } l\geqslant 0,\\ &p_{-l-1,l}^{(+)}(x)=0 \quad\text{and}\quad q_{0,l}^{(+)}(x)=1 \quad\text{for } l<0,\end{aligned} \tag{4.4}$$

we reach the limits

$$\lim_{x\to-\infty} e^{-j^3x}\begin{bmatrix} y_l^{(+)}(x) \\ z_l^{(+)}(x)\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \qquad \text{for } l\geqslant 0,$$
$$\lim_{x\to-\infty} e^{+j^3x}\begin{bmatrix} y_l^{(+)}(x) \\ z_l^{(+)}(x)\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \qquad \text{for } l< 0, \tag{4.5}$$

in agreement with the asymptotic solutions (3.5).

Proof. We perform the proof for $y_l^{(+)}(x)$ and $z_l^{(+)}(x)$ with $l\geqslant 0$; in the other cases one may proceed similarly. With the notations

$$p_{n,l}^{(+)}(x) = \sum_{\nu=0}^{2n} c_\nu^{(n,l)} x^\nu, \qquad q_{n+l,l}^{(+)}(x) = \sum_{\nu=0}^{2n-1} d_\nu^{(n,l)} x^\nu, \qquad n\geqslant 1, c_0^{(0,l)}=1, l\geqslant 0, \tag{4.6}$$

the polynomial degrees being deduced from recursion (4.3), this recursion can be written as the algebraic scheme

$$2nc_{2n}^{(n,l)} = \beta d_{2n-1}^{(n,l)}, \qquad 2(n+l)d_{2n-1}^{(n,l)} = -\beta c_{2n-2}^{(n-1,l)}, \qquad n\geqslant 1;$$
$$(2n-k+2)c_{2n-k+2}^{(n,l)} + 2nc_{2n-k+1}^{(n,l)} = -\lambda d_{2n-k+1}^{(n,l)} + \beta d_{2n-k}^{(n,l)}, \qquad 2\leqslant k\leqslant 2n, \qquad n\geqslant 1;$$
$$(2n-k+1)d_{2n-k+1}^{(n,l)} + 2(n+l)d_{2n-k}^{(n,l)} = \mu c_{2n-k}^{(n-1,l)} - \beta c_{2n-k-1}^{(n-1,l)}, \qquad 2\leqslant k\leqslant 2n-1, \quad n\geqslant 2;$$
$$c_1^{(n,l)} + 2nc_0^{(n,l)} = -\lambda d_0^{(n,l)}, \qquad d_1^{(n,l)} + 2(n+l)d_0^{(n,l)} = \mu c_0^{(n-1,l)}, \qquad n\geqslant 1. \tag{4.7}$$

This scheme starts with the coefficients

$$c_{2n}^{(n,l)} = (-1)^n(\tfrac{1}{2}\beta)^{2n}\frac{l!}{n!(n+l)!}, \qquad n\geqslant 0;$$
$$d_{2n-1}^{(n,l)} = (-1)^n(\tfrac{1}{2}\beta)^{2n-1}\frac{l!}{(n-1)!(n+l)!}, \qquad n\geqslant 1. \tag{4.8}$$

We introduce the ratios

$$|\lambda| = \xi|\beta|, \qquad |\mu| = \eta|\beta|;$$
$$|c_{2n-k}^{(n,l)}| = \rho_k^{(n,l)}|c_{2n}^{(n,l)}|, \qquad 0\leqslant k\leqslant 2n, \qquad n\geqslant 0;$$
$$|d_{2n-k-1}^{(n,l)}| = \sigma_k^{(n,l)}|d_{2n-1}^{(n,l)}|, \qquad 0\leqslant k\leqslant 2n-1, \qquad n\geqslant 1;$$
$$\rho_0^{(n,l)} = 1, \quad n\geqslant 0; \qquad \sigma_0^{(n,l)} = 1, \quad n\geqslant 1. \tag{4.9}$$

Then we arrive at the inequalities

$$\rho_{k-1}^{(n,l)} \leqslant \frac{2n-k+2}{2n}\rho_{k-2}^{(n,l)} + \xi\sigma_{k-2}^{(n,l)} + \frac{2n-k+1}{2(n+l)}\sigma_{k-2}^{(n,l)} + \eta\rho_{k-2}^{(n-1,l)} + \rho_{k-1}^{(n-1,l)},$$
$$\sigma_{k-1}^{(n,l)} \leqslant \frac{2n-k+1}{2(n+l)}\sigma_{k-2}^{(n,l)} + \eta\rho_{k-2}^{(n-1,l)} + \rho_{k-1}^{(n-1,l)}, \qquad 2\leqslant k\leqslant 2n-1, \quad n\geqslant 2;$$
$$\rho_{2n-1}^{(n,l)} \leqslant \frac{1}{n}\rho_{2n-2}^{(n,l)} + \xi\sigma_{2n-2}^{(n,l)} + \frac{1}{2(n+l)}\sigma_{2n-2}^{(n,l)} + \eta\rho_{2n-2}^{(n-1,l)},$$
$$\sigma_{2n-1}^{(n,l)} \leqslant \frac{1}{2(n+l)}\sigma_{2n-2}^{(n,l)} + \eta\rho_{2n-2}^{(n-1,l)}, \qquad \rho_{2n}^{(n,l)} \leqslant \frac{1}{2n}\rho_{2n-1}^{(n,l)} + \xi\sigma_{2n-1}^{(n,l)}, \qquad n\geqslant 1. \tag{4.10}$$

We simplify these inequalities using $(2n-k+2)/2n \leqslant 1$ and $(2n-k+1)/2(n+l) \leqslant 1$, and iterate them to obtain

$$
\begin{aligned}
&\rho_{k+1}^{(n,l)} \leqslant \sum_{\nu=\frac{1}{2}(k+2)}^{n} [\rho_k^{(\nu,l)} + (1+\xi)\sigma_k^{(\nu,l)} + \eta\rho_k^{(\nu-1,l)}], && k=0,2,\ldots,2n-2;\\
&\sigma_{k+1}^{(n,l)} \leqslant \sum_{\nu=\frac{1}{2}(k+2)}^{n} [\rho_k^{(\nu,l)} + (1+\xi)\sigma_k^{(\nu,l)} + \eta\rho_k^{(\nu-1,l)}], && k=0,2,\ldots,2n-4;\\
&\rho_{k+1}^{(n,l)} \leqslant \sum_{\nu=\frac{1}{2}(k+1)}^{n} [\rho_k^{(\nu,l)} + (1+\xi)\sigma_k^{(\nu,l)} + \eta\rho_k^{(\nu-1,l)}], && k=1,\ldots,2n-3;\\
&\sigma_{k+1}^{(n,l)} \leqslant \sum_{\nu=\frac{1}{2}(k+1)}^{n} [\rho_k^{(\nu,l)} + (1+\xi)\sigma_k^{(\nu,l)} + \eta\rho_k^{(\nu-1,l)}], && k=1,\ldots,2n-5;\\
&\sigma_{2n-2}^{(n,l)} \leqslant \rho_{2n-3}^{(n-1,l)} + \xi\sigma_{2n-3}^{(n-1,l)} + \sigma_{2n-3}^{(n,l)} + \eta\rho_{2n-3}^{(n-1,l)}, && n \geqslant 2;\\
&\sigma_{2n-1}^{(n,l)} \leqslant \sigma_{2n-2}^{(n,l)} + \eta\rho_{2n-2}^{(n-1,l)}, \qquad \rho_{2n}^{(n,l)} \leqslant \rho_{2n-1}^{(n,l)} + \xi\sigma_{2n-1}^{(n,l)}, && n \geqslant 1;
\end{aligned}
\tag{4.11}
$$

here we have defined $\rho_k^{(n,l)} = 0$ for $k > 2n$. Iterated insertion then leads to the result

$$
\begin{aligned}
&\rho_{k+1}^{(n,l)} \leqslant \alpha^{k+1} \sum_{n_1=\frac{1}{2}(k+1)}^{n} \sum_{n_2=\frac{1}{2}(k+1)}^{n_1} \sum_{n_3=\frac{1}{2}(k-1)}^{n_2} \sum_{n_4=\frac{1}{2}(k-1)}^{n_3} \cdots \sum_{n_k=1}^{n_{k-1}} \sum_{n_{k+1}=1}^{n_k} 1,\\
&\qquad k=1,3,5,\ldots,2n-1,\\
&\text{and}\\
&\sigma_{k+1}^{(n,l)} \leqslant \alpha^{k+1} \text{ (the same sum)}, \qquad k=1,3,5,\ldots,2n-3;\\
&\rho_{k+1}^{(n,l)} \leqslant \alpha^{k+1} \sum_{n_1=\frac{1}{2}(k+2)}^{n} \sum_{n_2=\frac{1}{2}k}^{n_1} \sum_{n_3=\frac{1}{2}k}^{n_2} \sum_{n_4=\frac{1}{2}(k-2)}^{n_3} \sum_{n_5=\frac{1}{2}(k-2)}^{n_4} \cdots \sum_{n_k=1}^{n_{k-1}} \sum_{n_{k+1}=1}^{n_k} 1,\\
&\qquad k=0,2,4,\ldots,2n-2,
\end{aligned}
\tag{4.12}
$$

and

$$\sigma_{k+1}^{(n,l)} \leqslant \alpha^{k+1} \text{ (the same sum)}, \qquad k=0,2,4,\ldots,2n-2; \quad \alpha = 2+\xi+\eta.$$

Obviously these sums are majorised by

$$\alpha^{k+1} \sum_{n_1=1}^{n} \sum_{n_2=1}^{n_1} \cdots \sum_{n_k=1}^{n_{k-1}} \sum_{n_{k+1}=1}^{n_k} 1.$$

By induction with respect to k we prove the identity

$$\sum_{n_1=1}^{n} \sum_{n_2=1}^{n_1} \cdots \sum_{n_k=1}^{n_{k-1}} \sum_{n_{k+1}=1}^{n_k} 1 = \binom{n+k}{k+1} = \frac{1}{(k+1)!} \sum_{n_1=1}^{n} \sum_{n_2=1}^{n+1} \sum_{n_3=1}^{n+2} \cdots \sum_{n_k=1}^{n+k-1} \sum_{n_{k+1}=1}^{n+k} 1. \tag{4.13}$$

Therefore we obtain

$$
\begin{aligned}
&\rho_{k+1}^{(n,l)} \leqslant \alpha^{k+1}\binom{n+k}{k+1}, && k=0,1,2,\ldots,2n-1;\\
&\sigma_{k+1}^{(n,l)} \leqslant \alpha^{k+1}\binom{n+k}{k+1}, && k=0,1,2,\ldots,2n-2.
\end{aligned}
\tag{4.14}
$$

Thus we get finally

$$\begin{aligned} |c_{2n-k}^{(n,l)}| &\leqslant (2\alpha|\beta|)^{2n}/(n!)^2, \qquad k=0,1,2,\ldots,2n; \\ |d_{2n-k-1}^{(n,l)}| &\leqslant (2\alpha|\beta|)^{2n-1}/[n!(n-1)!], \qquad k=0,1,2,\ldots,2n-1. \end{aligned} \tag{4.15}$$

This estimate allows some majorant which implies uniform convergence on each compact subset of the complex plane, and especially uniform convergence on the negative real line, of the series (4.2). This completes the proof.

The limit (4.5) shows that our series (4.2) represents the physically relevant solutions of equation (4.1). The expansion (4.2) can be generalised to an uniformly convergent series, which represents the general solution of equation (4.1) and exhibits the asymptotic behaviour (3.5) in the limit $x \to -\infty$.

5. Conclusion

Solutions of the Schrödinger and Dirac equations for electrons in logarithmic potentials have been given in terms of perturbation expansions, whose uniform convergence we have proved by an estimate of their coefficients. In this connection it seems worthwhile to examine the convergence behaviour of differential equations in the following general sense. Given a system of linear differential equations with exponential polynomial coefficients and entire solutions only, try to prove uniform convergence of an expansion in terms of exponential polynomials, analogously to corresponding theorems about the representation of holomorphic solutions as power series.

Concerning our asymptotic solutions derived from stability theorems for systems of linear differential equations, we shall try to understand this asymptotic behaviour as the classical limit of quantum mechanics in the sense of JWKB, especially with respect to the electron spin.

Our present efforts are devoted to the calculation of measured electron interferences (Möllenstedt and Düker 1956, Donati *et al* 1973, Merli *et al* 1976) by imposing appropriate boundary conditions in order to determine the experimentally cut off potential and the central wire.

Acknowledgments

We are indebted to H Mitter for initiating this work and stimulating remarks. We thank D Gronau, H Latal, L Reich, P Urban and P Vogl for fruitful discussions.

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